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# Unicity of Enrichment over $Cat$ or $Gpd$

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## Abstract

Enrichment of ordinary monads over  $Cat$  or  $Gpd$  is fundamental to Max Kelly's unified theory of coherence for categories with structure. So here, we investigate existence and unicity of enrichments of ordinary functors, natural transformations, and hence also monads, over  $Cat$  and  $Gpd$ . We show that every ordinary natural transformation between 2-functors whose domain 2-category has either tensors or cotensors with the arrow category is 2-natural. We use that to prove that an ordinary monad, or endofunctor, on such a 2-category has at most one enrichment over  $Cat$  or  $Gpd$ . We also describe a monad on  $Cat$  that has no enrichment. So enrichment over  $Cat$  is a non-trivial property of a monad rather than a structure that is additional to it. Finally, we present an example, due to Kelly, of  $V$  other than  $Cat$  or  $Gpd$  and an ordinary monad for which more than one enrichment over  $V$  exists, showing that our main theorem is specific to  $Cat$  and  $Gpd$ .

## 1 Introduction

A central theme, perhaps the central theme, of Max Kelly's research was the study of coherence for categories with algebraic structure. His most prominent paper on the topic was [1], complementing his definitive book on enriched categories [4]. My own scientific career has been founded on my four years, 1984-88, as his research fellow working on the topic, a fact for which I am profoundly grateful. Max's ideas have now become embedded into theoretical computer science (see [7] for an overview and [8] for a recent application of them).

In the Australian summer of 1991/92, Max invited me to Sydney to continue our work on coherence for categories with algebraic structure. He asked the following question: given an ordinary monad  $T$  on  $Cat$ , could there be at most one enrichment of  $T$  over  $Cat$ ? A variant of the question, cohering with his research on coherence, was whether an ordinary monad on  $Cat$  can have at most one enrichment over the cartesian closed category  $Gpd$  of groupoids.

We knew that there are monads on  $Cat$  that admitted no enrichment: an example appears in Section 3. But is there at most one? The question was a natural one for Max to ask, coming after his proof, with Foltz and Lair, that  $Cat$  possesses precisely two symmetric monoidal closed structures [2], and coming before his investigation with Lack of property-like structures on  $Cat$  [6]. A positive answer would indicate that 2-monadicity is a property of a monad on  $Cat$  rather than a genuine additional structure.

In fact, the answer is Yes, the answer following from an existence result with unicity implicit: if a 2-category  $A$  admits tensors or cotensors with the arrow category (see [5]), given any pair of 2-functors of the form  $F, G : A \longrightarrow B$ , any ordinary natural transformation  $\alpha : F_0 \Rightarrow G_0 : A_0 \longrightarrow B_0$  is necessarily 2-natural. We prove that result in Section 2.

Having proved the above result, we proceed to Max's uniqueness question. The answer to his question can be framed in a more general setting than that in which he originally asked it: given any 2-category  $A$  satisfying a very mild cocompleteness or completeness condition, namely the existence of either tensors or cotensors with the arrow category, any monad on  $A$  possesses at most one enrichment over  $Cat$ , with the same applying for enrichment over  $Gpd$ . Note that there is no assumption of rank on the monad, and the base 2-category need not be either complete or cocomplete. We deduce the result in Section 3.

Inevitably, having proved the result, the question arises whether the same is true for enrichment over a more general class of symmetric monoidal closed categories  $V$ . So, in Section 4, we present a counter-example, due to Max, to show that the argument does not extend, i.e., we give a  $V$  other than  $Cat$  or  $Gpd$  together with a monad with two possible enrichments over  $V$ .

## 2 Existence of Enrichment over $Cat$ or $Gpd$ of Ordinary Natural Transformations

In this section, we show that if a 2-category  $A$  admits tensors or cotensors with the arrow category  $\rightarrow$ , then for any 2-functors  $F, G : A \longrightarrow B$ , any natural transformation

$$\alpha : F_0 \Rightarrow G_0 : A_0 \longrightarrow B_0$$

is 2-natural.

We first observe that, in order to prove the result, we do need a condition on  $A$ , i.e., if we drop the assumption on  $A$  that it admits either tensors or cotensors with  $\rightarrow$ , there may exist natural transformations  $\alpha$  that are not 2-natural.

**Example 2.1** *Let  $A$  be the free 2-category on a 2-cell. So  $A$  has two objects, which we denote by 0 and 1, two 1-cells  $x, y : 0 \longrightarrow 1$ , and a 2-cell  $\gamma : x \Rightarrow y$ , but no other non-trivial 1-cells or 2-cells.*

*Let  $\parallel$  denote the category with two objects  $X$  and  $Y$  and two maps  $f, g : X \longrightarrow Y$ , i.e.,  $\parallel$  is the free category on a pair of parallel maps.*

Define  $F, G : A \longrightarrow \text{Cat}$  by putting

$$\begin{aligned} F0 &= G0 = 1 & F1 &= G1 = \parallel \\ Fx &= Gx = X & Fy &= Gy = Y \\ F\gamma &= f & G\gamma &= g \end{aligned}$$

Then  $F_0 = G_0$ , so the identity is a natural transformation from  $F_0$  to  $G_0$ . But the identity is not a 2-natural transformation as  $F$  does not equal  $G$ .

We now proceed to the main result of the section via some lemmas.

**Lemma 2.2** [The main lemma] Given any categories  $C$  and  $D$ , consider functors  $f : C \longrightarrow D$  and  $g : C^\rightarrow \longrightarrow D^\rightarrow$  such that the following three diagrams in  $\text{Cat}$  commute:

$$\begin{array}{ccc} \begin{array}{ccc} C^\rightarrow & \xrightarrow{g} & D^\rightarrow \\ \text{dom} \downarrow & & \downarrow \text{dom} \\ C & \xrightarrow{f} & D \end{array} & \begin{array}{ccc} C^\rightarrow & \xrightarrow{g} & D^\rightarrow \\ \text{cod} \downarrow & & \downarrow \text{cod} \\ C & \xrightarrow{f} & D \end{array} & \begin{array}{ccc} C^\rightarrow & \xrightarrow{g} & D^\rightarrow \\ \delta \uparrow & & \uparrow \delta \\ C & \xrightarrow{f} & D \end{array} \end{array}$$

Then  $g = f^\rightarrow$ .

**Proof** Consider the square

$$\begin{array}{ccc} x & \xrightarrow{\rho} & y \\ \alpha \downarrow & & \downarrow \beta \\ u & \xrightarrow{\sigma} & v \end{array} \quad (1)$$

as an arrow  $(\rho, \sigma) : \alpha \longrightarrow \beta$  in  $C^\rightarrow$ . The commutativities above involving  $\text{dom}$  and  $\text{cod}$  ensure that  $g(\rho, \sigma) : g\alpha \longrightarrow g\beta$  has the form

$$\begin{array}{ccc} fx & \xrightarrow{f\rho} & fy \\ g\alpha \downarrow & & \downarrow g\beta \\ fu & \xrightarrow{f\sigma} & fv \end{array} \quad (2)$$

The commutativity involving  $\delta$  ensures that

$$g1_x = 1_{fx} \quad (3)$$

Now, take for Diagram 1 the diagram

$$\begin{array}{ccc} x & \xrightarrow{\rho} & y \\ \rho \downarrow & & \downarrow id \\ y & \xrightarrow{id} & y \end{array}$$

In the light of Equation 3, Diagram 2 yields commutativity of the diagram:

$$\begin{array}{ccc} fx & \xrightarrow{f\rho} & fy \\ g\rho \downarrow & & \downarrow id \\ fy & \xrightarrow{id} & fy \end{array}$$

so  $g\rho = f\rho$ , so  $g = f^{\rightarrow}$  as desired.

**Proposition 2.3** *Let  $F, G : A \rightarrow B$  be 2-functors, where the 2-categories  $A$  and  $B$  admit tensors with the arrow category  $\rightarrow$ . Then, for any natural transformation  $\alpha : F_0 \Rightarrow G_0$ , the diagram*

$$\begin{array}{ccc} \rightarrow \otimes FX & \xrightarrow{\rightarrow \otimes \alpha_X} & \rightarrow \otimes GX \\ (\tau_F)_X \downarrow & & \downarrow (\tau_G)_X \\ F(\rightarrow \otimes X) & \xrightarrow{\alpha_{\rightarrow \otimes X}} & G(\rightarrow \otimes X) \end{array} \quad (4)$$

*commutes, where  $\tau_F$  and  $\tau_G$  are the canonical comparison maps induced by the 2-functoriality of  $F$  and  $G$  respectively.*

**Proof** Given any 0-cell  $Y$  in  $B$ , consider the two composites

$$A(\rightarrow \otimes X, Y) \xrightarrow{G} B(G(\rightarrow \otimes X), GY) \xrightarrow{B(f, GY)} B(\rightarrow \otimes FX, GY)$$

generated by the two maps from  $\rightarrow \otimes FX$  to  $G(\rightarrow \otimes X)$  given by Diagram 4. By the enriched Yoneda lemma, the two legs of Diagram 4 are equal if and only if these two composites are equal. But pre- and post-composing the two composites by the canonical isomorphisms defining tensor yield functors of the form

$$A(X, Y)^{\rightarrow} \rightarrow B(FX, GY)^{\rightarrow}$$

which, by routine application of naturality and functoriality, satisfy the coherence conditions of Lemma 2.2 relative to the functor

$$A(X, Y) \xrightarrow{G} B(GX, GY) \xrightarrow{B(\alpha_X, GY)} B(FX, GY)$$

So by Lemma 2.2, the two composites are equal, and hence Diagram 4 commutes.

**Theorem 2.4** *If  $A$  admits tensors with  $\rightarrow$  and  $F, G : A \rightarrow B$  are 2-functors, any ordinary natural transformation  $\alpha : F_0 \Rightarrow G_0$  is 2-natural.*

**Proof** By the 2-categorical Yoneda lemma, we may assume without loss of generality that  $B$  has tensors with  $\rightarrow$ .

To give a 2-cell  $\lambda : h_0 \Rightarrow h_1 : X \rightarrow Y$  in a 2-category that admits tensors with  $\rightarrow$  is equivalent to giving a map  $h$  from  $\rightarrow \otimes X$  to  $Y$ . Post-composition with Diagram 4 together with routine diagram manipulation yields the result.

The proof of Theorem 2.4 can also be seen as an instance of general theory about the relationship between enriched categories and categories with actions [3]. By duality, Theorem 2.4 can also be written as follows:

**Corollary 2.5** *Given a 2-category  $A$  that has cotensors with  $\rightarrow$ , and 2-functors  $F, G : A \rightarrow B$ , any ordinary natural transformation  $\alpha : F_0 \Rightarrow G_0$  is 2-natural.*

### 3 Unicity of Enrichment over $Cat$ or $Gpd$

The main goal of this paper is to show that, under a very mild condition on a base 2-category, every ordinary monad on it has at most one enrichment. But, for completeness, we first show that there exist ordinary monads on  $Cat$  that do not admit any enrichment at all, even over the cartesian closed category  $Gpd$  of groupoids.

**Example 3.1** *Consider the endofunctor  $T$  on  $Cat$  given by*

$$T(C) = C + (N \times ob(C))$$

*where  $N$  is the discrete category on the natural numbers. The action of  $T$  on maps is evident. The functor  $T$  is finitary and is the functor part of the monad on  $Cat$  for which an algebra is a category  $C$  together with a function  $E : ob(C) \rightarrow ob(C)$ .*

*But  $T$  cannot be extended to 2-cells, even to invertible ones. For instance, putting  $C = 1$ , taking  $D$  to be  $Iso$ , the two object category that is equivalent to  $1$ , and letting  $f$  and  $g$  be the two distinct functors from  $C$  to  $D$ , it follows that  $f$  is isomorphic to  $g$  but  $Tf$  is not isomorphic to  $Tg$ . So there is no possible enrichment of the ordinary functor  $T$  over either  $Cat$  or  $Gpd$ .*

We now proceed towards showing that if a 2-category  $A$  admits tensors or cotensors with the arrow category  $\rightarrow$ , then any ordinary monad  $S$  on  $A_0$  admits at most one enrichment over  $Cat$ .

**Definition 3.2** *Given 2-categories  $A$  and  $B$  and an ordinary functor  $H : A_0 \longrightarrow B_0$ , an enrichment of  $H$  is any 2-functor  $K : A \longrightarrow B$  for which  $K_0 = H$ . And given an ordinary monad  $S$  on  $A_0$ , an enrichment of  $S$  is any 2-monad  $T$  on  $A$  for which  $T_0 = S$ .*

**Corollary 3.3** *If  $A$  has tensors or cotensors with  $\rightarrow$ , then any ordinary functor of the form  $H : A_0 \longrightarrow B_0$  has at most one enrichment.*

**Proof** Let  $F, G : A \longrightarrow B$  be 2-functors for which  $F_0 = G_0 : A_0 \longrightarrow B_0$ . Then the identity is an ordinary natural transformation from  $F_0$  to  $G_0$ . So by Theorem 2.4 and Corollary 2.5, it is 2-natural, and so  $F = G$ .

**Corollary 3.4** *If  $A$  has tensors or cotensors with  $\rightarrow$ , then any ordinary monad on  $A_0$  has at most one enrichment. Moreover, an enrichment exists if and only if its underlying functor enriches.*

**Proof** Suppose  $S$  and  $T$  are 2-monads on  $A$  for which  $S_0 = T_0$ . By Corollary 3.3,  $S$  and  $T$  have the same underlying 2-functors. The unit and multiplication of  $S$  and  $T$  agree as ordinary natural transformations, so must agree as 2-natural transformations. So  $S = T$ . The final claim follows immediately from Theorem 2.4 and Corollary 2.5.

The results of this section all extend trivially from *Cat* to *Gpd*, the cartesian closed category of groupoids because every *Gpd*-category is, a fortiori, a 2-category, and a 2-functor between *Gpd*-categories is precisely a *Gpd*-functor between them.

## 4 Lack of Unicity of Enrichment for General $V$

The main result of this paper asserts that an ordinary monad on a 2-category, subject to a very mild condition on the 2-category, has at most one enrichment over *Cat*. The corresponding fact trivially holds in regard to enrichment over any  $V$  for which the functor  $V_0(I, -) : V_0 \longrightarrow \text{Set}$  is faithful, e.g., for  $V = \text{Ab}$ , *Poset* or  *$\omega\text{Cpo}$* . But it fails for  $V = \text{Gr-Ab}$  and  $V = \text{D-Gr-Ab}$ , hence for some  $R$  with  $V = \text{D-Gr-R-Mod}$  [4].

**Example 4.1** *Consider  $V = \text{D-Gr-R-Mod}$  even for  $R$  being the ring  $\mathbb{Z}$  of integers. There are representable  $V$ -functors  $[-, A], [-, B] : V^{op} \longrightarrow V$  together with an ordinary natural transformation from  $[-, A]$  to  $[-, B]$  that is not  $V$ -natural, or, equivalently by the enriched Yoneda lemma, that does not arise uniquely from a map from  $A$  to  $B$ .*

*Define  $A$  by putting  $A_n = \mathbb{Z}$  if  $n = 0$  and 0 otherwise. Define  $B$  by putting  $B_n = \mathbb{Z}$  if  $n = 1$  and 0 otherwise. This is sufficient to determine the rest of the data.*

*Observe that the zero map is the only map from  $A$  to  $B$ . Letting  $C^{-n}(X)$  denote the set of  $n$ -cochains of  $X$ , one has*

$$[X, A]_n = [X_n, Z] = C^{-n}(X)$$

and

$$[X, B]_n = [X_{-n+1}, Z] = C^{-n+1}(X)$$

Consider the family of maps

$$\alpha_X : [X, A] \longrightarrow [X, B]$$

given by

$$C^{-n}(X) \xrightarrow{\delta} C^{-n+1}(X)$$

where delta is defined by composition with  $\partial : X_{n+1} \longrightarrow X_n$ .

Observe that  $\delta$  is a chain complex map since  $\delta\delta = \delta\delta = 0$ . Moreover, it is natural in  $X$  because

$$\begin{array}{ccc} [X, A] & \xrightarrow{\delta} & [X, B] \\ [g, A] \downarrow & & \downarrow [g, B] \\ [Y, A] & \xrightarrow{\delta} & [Y, B] \end{array}$$

commutes whenever  $\partial g = g\partial$ , i.e., whenever  $g$  is a chain map.

But it is non-zero, yielding the desired counterexample.

Example 4.1 shows that the analysis of this paper is specific to *Cat* and *Gpd*. Evidently, one could extend it in very specific ways, but the proof is remarkably special, and it is certainly does not extend to those  $V$  that have been oft-studied to date.

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